Combinatorics of sections of polytopes and Coxeter groups in Lobachevsky spaces

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Coxeter classified all discrete isometry groups generated by reflections that act on a Euclidean space or on a sphere of an arbitrary dimension (see [1]). His fundamental work became classical long ago. Lobachevsky spaces (classical hyperbolic spaces) are as symmetric as Euclidean spaces and spheres. However, discrete isometry groups generated by reflections, with fundamental polytopes of finite volume (see [2]), are not classified for Lobachevsky spaces. In 1985, M.N. Prokhorov and myself proved the following theorem.

Theorem 0.1 [3, 4] *In a Lobachevsky space of dimension > 995 there are no discrete isometry groups generated by reflections, with fundamental polytope of finite volume.*

For groups with compact fundamental polytopes, an analogous result had been previously obtained by E.B. Vinberg ([5, 6]). According to his theorem, such groups do not exist in Lobachevsky spaces of dimensions > 29. (Most likely, it is possible to reduce the number 29 considerably, all the more so the number 995. But nobody knows how to do that). The result of E.B. Vinberg came after the works of V.V. Nikulin ([7, 8]) who worked out the case of arithmetic groups. Nikulin estimated the average number of l-dimensional faces on k-dimensional faces of simple n-dimensional polytopes (the definition of a simple polytope is given below in this section) and applied his estimate to groups generated by reflections. In fact, a compact fundamental polytope of such a group is always simple, and Nikulin's estimate is applicable to it. Prokhorov and myself followed Nikulin's plan. We performed non-overlapping parts of the necessary work for the realization of this plan. Namely, Prokhorov proved the following theorem.

Theorem 0.2 [4] *In a Lobachevsky space of dimension > 995, there are no discrete groups generated by reflections, with fundamental polytope of finite volume, satisfying Nikulin's estimate.*

It is known that if a polytope of finite volume is the fundamental polytope of a group generated by reflections in a Lobachevsky space, then this polytope is always almost simple, and, therefore, is simple at the edges (the definitions of almost simple polytopes and polytopes simple at the edges are given below in this section). I proved the following theorem.

Theorem 0.3 [3] *Nikulin's estimate holds for polytopes simple at the edges.*

Corollary 0.4 [3] *Nikulin's estimate holds for almost simple polytopes.*

Theorem 0.1 follows from Theorem 0.2 and Corollary 0.4. Nikulin proved his estimate using a very hard theorem — the theorem on the h-vector — a necessary and sufficient condition on a collection of integers to be the h-vector of a simple polytope (a variant of Nikulin's proof is given in Section 2). R. Stanley's proof of the necessity part of the theorem on the h-vector uses nontrivial results from algebraic geometry (see Section 3 for his proof). During the time passed from [3], several elementary proofs of the theorem on the h-vector were found (see [9], [10]). But they are also far from being simple.

In [3], I found a simple elementary proof of the statement from Corollary 0.4 (see the remark in Section 9 after Theorem 9.1). This statement is necessary for the proof of Theorem 0.1. It contains Nikulin's original estimate as a partial case. But I failed to find a simple proof of Theorem 0.3 — my proof of it is based on the theorem on the h-vector.

The *Klein model* of a Lobachevsky space is the interior U of the unit ball in a Euclidean space. Polytopes in this model are intersections of Euclidean polytopes with the region U. If a polytope is bounded in the Lobachevsky space, then in the Klein model it lies entirely in the region U. If a polytope in the Lobachevsky space has a finite volume, then in the Klein model it can intersect the horizon ∂U by vertices only. Recall the definition of a *simple polytope* and of a *polytope simple at the edges*. These definitions play a central role in this article.

Definition 1 A convex n-dimensional polytope is said to be *simple*, if its every vertex is incident to exactly n facets.

A neighborhood of any vertex of a simple n-dimensional polytope can be transformed into a neighborhood of the origin in the positive n-dimensional octant $(R^+)^n$ by an affine transformation. Hence *exactly n edges meet at each vertex of a simple n-dimensional polytope*.

Definition 2 A convex n-dimensional polytope is said to be *simple at the edges*, if its every edge is incident to exactly $(n-1)$ facets.

Let us also give the definition of an almost simple polytope.

Definition 3 A convex n-dimensional polytope is said to be *almost simple*, if it looks like the cone over a product of simplices at its every vertex.

It is clear that each simple polytope is almost simple, and each almost simple polytope is simple at the edges. At its every vertex, a simple n-dimensional polytope looks like the cone over an $(n-1)$-dimensional simplex. At its every vertex, an n-dimensional polytope simple at the edges looks like the cone over an $(n-1)$-dimensional simple polytope.

While preparing my talk for the "Coxeter's legacy" conference, I found out that my article [3], which had been written 19 years ago, is hard to read. The reason is

that the journal "Functional Analysis and its Applications", where I published my article, had a restricted space. That is why I had to abridge the article considerably. Luckily, I found an unabridged variant of the article, which helped me a lot in preparation of my talk and in writing this article.

My student V.A. Timorin found another proof of Theorem 0.3 (see [11]). His arguments are parallel to mine for the most part, but they do not at all use the combinatorics of sections of polytopes, which my proof relies upon. On one hand, this shortens the proof essentially. On the other hand, the facts from article [3] related to combinatorics of sections of polytopes, are interesting by themselves. They explain the geometric meaning of Nikulin's estimate.

This article is devoted to combinatorics of sections of polytopes and to a generalization of Nikulin's estimate. It is a considerably expanded and revised version of article [3].

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1 Statements of Nikulin's theorem and of the theorem on the h-vector

Nikulin's estimate deals with the *average number of l-dimensional faces on k-dimensional faces of an n-dimensional polytope.* This average number is defined as follows. First, for every k-dimensional face (k-face for short) of the polytope, we compute the number of all l-faces on it. Then we take the arithmetic mean of these numbers over all k-faces of the polytope. By the *total number of l-faces on k-faces* we mean the number of pairs consisting of a k-face and an l-face of it. Thus the average number of l-faces on k-faces equals to the total number of l-faces on k-faces divided by the number of k-faces.

Let us start with 3-dimensional case. The following classical theorem is well known.

Theorem 1.1 *The average number of edges on faces of a convex 3-polytope is strictly less than 6.*

Before proving this theorem, let us discuss the simplest example.

Example 1 Consider a prism whose base is a convex n-gon. The upper and the lower bases of the prism contain n edges each. Its side faces are n quadrangles. Hence the total number of edges on faces of the prism is $n+n+4n = 6n$. The prism has $(n+2)$ faces. Hence the average number of edges on its faces is $6n/(n+2) < 6$.

The example given above shows that the estimate from the theorem can not be improved, i.e. that the number 6 in its statement can not be replaced with a smaller number.

Proposition 1.2 *The estimate from the theorem holds for simple 3-polytopes.*

Proof Denote by f_0, f_1 and f_2 the number of vertices, edges and faces of the polytope, respectively. We have

$$f_0 - f_1 + f_2 = 2,$$

$$3f_0 = 2f_1.$$

The first of these identities is the Euler formula. The second identity follows from the fact that exactly 3 edges meet at each vertex of the polytope. ¿From these equalities we obtain that $2f_1/f_2 = 6 - 12/f_2$. The claim is thus proved, since the total number of edges on faces of the polytope equals to $2f_1$, and the number f_2 of its faces is positive. $\square$

Theorem 1.1 can be proved in the same way as Proposition 1.2. We only need to replace the equality $3f_0 = 2f_1$ with the inequality $3f_0 \leq 2f_1$. The latter means that at least 3 edges meet at each vertex of the polytope. Using the Euler formula, we obtain the inequality $2f_1/f_2 \leq 6 - 12/f_2$ that implies the theorem. Nikulin generalized Proposition 1.2 to the multidimensional case. Namely, he proved the following theorem.

Nikulin's theorem *The average number of l-dimensional faces on k-dimensional faces of a simple n-dimensional polytope is strictly less than*

$$\binom{n-l}{n-k} \frac{\binom{[n/2]}{l} + \binom{[(n+1)/2]}{l}}{\binom{[n/2]}{k} + \binom{[(n+1)/2]}{k}}$$

for $0 \leq l < k \leq (n+1)/2$, $1 < k$.

According to Theorem 0.3, Nikulin's estimate holds for polytopes simple at the edges. Theorem 0.3 includes Theorem 1.1, since any convex 3-polytope is simple at the edges. If in the proof of Theorem 0.3 we confine ourselves with the 3-dimensional case, then we obtain a proof of Theorem 1.1 not using the Euler formula (see Section 10). The following statement is a supplement to Nikulin's theorem:

Proposition 1.3 *1) For each triple of integers l, k, n such that $0 \leq l < k$, $(n+1)/2 < k \leq n$, $1 < n$, the average number of l-dimensional faces on k-dimensional faces of a simple n-polytope can be arbitrarily large.*

2) For each triple of integers l, k, n satisfying the conditions of Nikulin's theorem, his estimate is best possible.

Here are the simplest examples.

Example 2 Examples For the triple $l = 0, k = n = 2$, the first part of the claim is obvious, since there exist convex polygons with any number of vertices. For the triple $l = 1, k = 2, n = 3$, Nikulin's estimate gives the number 6. As the example discussed above shows, this estimate is best possible.

Nikulin's estimate is rather cumbersome. Below, its geometric meaning is discussed (see Section 6). For now, let me just give the following remark.

Remark 1 For fixed integers l and k, as $n \to \infty$, Nikulin's estimate tends to the number $2^{k-l}\binom{k}{l}$, which is equal to the number of l-dimensional faces of the k-dimensional cube. Since the n-dimensional cube is a simple polytope for every n, and its k-dimensional faces are cubes, Nikulin's estimate is asymptotically exact.

The proof of Nikulin's theorem is based on the theory of simple polytopes, which is closely related to the theory of toric varieties. Recall the classical description of the f-vectors of simple polytopes. For a convex n-dimensional polytope Δ, the f-*vector* is the integer vector $(f_0, \ldots, f_n)$, whose component f_i equals to the number of i-faces of the polytope Δ for each $0 \leq i \leq n$ (in particular, $f_n = 1$). The polynomial $f(t) = f_0 + f_1 t + \cdots + f_n t^n$ is called the f-*polynomial* of the polytope Δ. The polynomial $h(t) = f(t - 1)$ is called the h-*polynomial* of the polytope Δ. The vector $(h_0, \ldots, h_n)$, whose components are the coefficients of the polynomial h (i.e.

$h(t) = h_0 + h_1 t + \cdots + h_n t^n$), is called the *h-vector* of the polytope Δ. The identity $f(t) = h(t+1)$ shows that *the h-vector determines the f-vector*, Namely, for every $0 \leq m \leq n$, we have $f_m = \sum\limits_{0 \leq i \leq n} \binom{i}{m} h_i$.

What integer vectors are the *h*-vectors of simple *n*-dimensional polytopes? A complete answer to this question is given by the following remarkable theorem.

Theorem on the *h*-vector (McMullen, Stanley, Billera, Lee) *For every simple n-dimensional polytope, the components $h_0, \ldots, h_n$ of its h-vector satisfy the following conditions:*

1. *(Dehn–Sommerville duality) For each $0 \leq i \leq n$, we have*

$$h_i = h_{n-i};$$

2. *All components of the h-vector are nonnegative, and the numbers h_0 and h_n are 1;*

3. *The h-vector is unimodal, i.e. $1 = h_0 \leq \cdots \leq h_{[n/2]}$;*

4. *The sequence of numbers $h_1 - h_0, h_2 - h_1, \ldots, h_{[n/2]} - h_{[n/2]-1}$ has a bounded rate of growth: for $i = 0, \ldots, [n/2] - 1$, we have the inequalities $h_{i+1} - h_i < Q_i(h_i - h_{i-1})$, where Q_i are some explicit functions of an integer argument. Functions Q_i are not simple, but we will not need their explicit form.*

For each integer vector $h = (h_0, \ldots, h_n)$ satisfying conditions 1)–4), there exists a simple n-dimensional polytope, whose h-vector is equal to h.

The Dehn–Sommerville duality was discovered in the beginning of the last century (see [12]). In its entire form, the theorem on the *h*-vector was first conjectured by McMullen (see [13]). Stanley proved the necessity of McMullen's conditions on the *h*-vector (see [14]). Stanley's proof is based on a nontrivial technique from algebraic geometry (see [14] and Section 3). For every integer vector *h* satisfying conditions 1)–4), Billera and Lee gave an example of a simple polytope, whose *h*-vector equals to *h*. Thus they concluded the proof of the theorem on the *h*-vector (see [15]).

Nikulin's estimate is a direct corollary from the theorem on the *h*-vector (see Section 2). In fact, to deduce this estimate, we only need parts 1) and 2) of the theorem on the *h*-vector, together with a couple of elementary lemmas given in Section 2.

Parts 1) and 2) of the theorem on the *h*-vector can be easily proved using Morse theoretic type argument. Namely, one can use a generic linear function on the polytope (see Section 4). Thus one obtains a simple proof of Nikulin's estimate. Similar arguments can be also employed to prove a generalization of Nikulin's estimate necessary for the Lobachevsky geometry. But this generalization makes use of the theorem on the *h*-vector in corpore (to be more precise, we will need part 3) of this theorem, which is the most difficult).

A simple argument based on a generic linear function on the polytope came to my mind when I was thinking about Stanley's proof. In Section 3, we will discuss the idea of Stanley's proof.

2 Derivation of Nikulin's estimate from the theorem on the *h*-vector

To deduce Nikulin's estimate from the theorem on the *h*-vector, we will need Lemmas 2.1 and 2.4. They are quite elementary. Lemma 2.1 is very intuitive. The proof of Lemma 2.4 is simple, but a little cumbersome. It is based on Claims 2.3 and 2.4

Let $A = (A_1, \ldots, A_m)$ and $B = (B_1, \ldots, B_m)$ be fixed vectors from $\mathbb{R}^m$, and suppose that all components of the vector A are strictly positive. Consider the set $\Omega \subset R^m$ defined by the relations $\alpha \in \Omega \Leftrightarrow \alpha_1 \geq 0, \ldots, \alpha_m \geq 0$ and $\alpha_1 + \cdots + \alpha_m > 0$. Denote by $\langle x, y \rangle$ the standard inner product of vectors $x, y \in \mathbb{R}^m$.

Lemma 2.1 *The maximum C of the function $F(\alpha) = \frac{\langle \alpha, B \rangle}{\langle \alpha, A \rangle}$ on the region Ω, $\alpha \in \Omega$, is equal to*

$$\max_{1 \leq i \leq m} \frac{B_i}{A_i}.$$

Furthermore, the maximum C is attained on any vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that its j-th components α_j vanish for all indices j such that $\frac{A_j}{B_j} < C$.

Proof Let Ω_1 be a subset of Ω defined by the condition $\alpha_1 + \cdots + \alpha_m = 1$. Multiplying the vector α by a positive number, we can arrange that $\alpha \in \Omega_1$. With the vector α, associate the point $(\langle \alpha, A \rangle, \langle \alpha, B \rangle)$ in the plane with coordinates a, b. The image of the set Ω_1 under this correspondence coincides with the convex polygon Δ that is the convex hull of the points (A_i, B_i), $i = 1, \ldots, m$. The polygon Δ lies in the right half-plane $a > 0$. The function $\frac{b}{a}$ on this half-plane is continuous, its level sets are rays beginning at 0. Furthermore, this function depends monotonely on the angle between this ray and the positive ray on the a-axis. The lemma is now geometrically evident. $\qquad \square$

Proposition 2.2 *1) Let A_1, A_2 be positive numbers, and B_1, B_2, μ_1, μ_2 non-negative numbers. Suppose that $\frac{B_1}{A_1} < \frac{B_2}{A_2}$, $0 < \mu_1 + \mu_2$, $\mu_1 \leq \mu_2$. Then*

$$\frac{B_1 + B_2}{A_1 + A_2} \leq \frac{\mu_1 B_1 + \mu_2 B_2}{\mu_1 A_1 + \mu_2 A_2}.$$

2) Assume additionally that $B_1 < B_2$ and that there are numbers λ_1 and λ_2 such that $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$ and $\lambda_1 < \mu_1 < \mu_2 < \lambda_2$. Then

$$\frac{B_1 + B_2}{A_1 + A_2} < \frac{\lambda_1 B_1 + \lambda_2 B_2}{\mu_1 A_1 + \mu_2 A_2}.$$

Proof Part 1) of Proposition 2.2 follows from Lemma 2.1. Indeed, according to Lemma 2.1, $\frac{B_1 + B_2}{A_1 + A_2} < \frac{B_2}{A_2}$. By the same Lemma 2.1,

$$\frac{B_1 + B_2}{A_1 + A_2} \leq \frac{\mu_1(B_1 + B_2) + (\mu_2 - \mu_1)B_2}{\mu_1(A_1 + A_2) + (\mu_2 - \mu_1)A_2} = \frac{\mu_1 B_1 + \mu_2 B_2}{\mu_1 A_1 + \mu_2 A_2}.$$

Part 2) follows from part 1). Indeed, since $\lambda_2 - \mu_2 \geq \mu_1 - \lambda_1 > 0$, we have $(\lambda_2 - \mu_2)B_2 \geq (\mu_1 - \lambda_1)B_2 > (\mu_1 - \lambda_1)B_1$. Hence $\lambda_1 B_1 + \lambda_2 B_2 > \mu_1 B_1 + \mu_2 B_2$. It remains to use the inequality from part 1). $\qquad \square$

For i and j such that $0 \leq i \leq n$ and $0 \leq j \leq (n+1)/2$, denote by $\varphi(j, i)$ the number

$$\frac{\binom{i}{j} + \binom{n-i}{j}}{\binom{[n/2]}{j} + \binom{[(n+1)/2]}{j}}.$$

Proposition 2.3 *1) The numbers $\varphi(0, i)$, $\varphi(1, i)$ and the numbers $\varphi(j, [n/2])$, $\varphi(j, [(n+1)/2])$ equal to 1;*

2) We have $\varphi(j, i) = \varphi(j, n - i)$;

3) For a fixed i such that $0 \leq i < [n/2]$, the numbers $\varphi(j, i)$ increase strictly as j runs from 1 to $[(n+1)/2]$.

Parts 1) and 2) are obvious. Let us prove part 3). We need to verify that for $0 \le i < [n/2]$ and $1 \le j \le (n-1)/2$, we have the inequalities $\varphi(j,i) < \varphi(j+1,i)$. This is easy to do using part 2) of Proposition 2.2. It suffices to set $B_1 = \binom{i}{j}$, $B_2 = \binom{n-i}{j}$, $A_1 = \binom{[n/2]}{j}$, $A_2 = \binom{[(n+1)/2]}{j}$. The following inequality holds:

$$\frac{\binom{i}{j}}{\binom{[n/2]}{j}} < \frac{\binom{n-i}{j}}{\binom{[(n+1)/2]}{j}}.$$

Indeed, the left hand side of this inequality is less than 1, but the right hand side is greater than 1. Furthermore, we have

$(j+1)\binom{i}{j+1} = \lambda_1 \binom{i}{j}$, where $\lambda_1 = \min(i-j,0)$;

$(j+1)\binom{n-i}{j+1} = \lambda_2 \binom{n-i}{j}$, where $\lambda_2 = n-i-j$;

$(j+1)\binom{[n/2]}{j+1} = \mu_1 \binom{[n/2]}{j}$, where $\mu_1 = [n/2]-j$;

$(j+1)\binom{[(n+1)/2]}{j+1} = \mu_2 \binom{[(n+1)/2]}{j}$, where $\mu_2 = [(n+1)/2]-j$.

The conditions of part 2) of Proposition 2.2 are satisfied, since $\lambda_1 + \lambda_2 \ge (n-2j) = \mu_1 + \mu_2$ and $(n-i-j) > [(n+1)/2]-j \ge [n/2]-j > \min(i-j,0)$. Using part 2) of Proposition 2.2, we obtain the desired inequality.

Lemma 2.4 *Let $0 \le l < k \le (n+1)/2$. For each $0 \le i \le n$, set $A_i = \binom{i}{k} + \binom{n-i}{k}$, $B_i = \binom{i}{l} + \binom{n-i}{l}$. Then*

$$\max_i \frac{B_i}{A_i} = \frac{\binom{[n/2]}{l} + \binom{[(n+1)/2]}{l}}{\binom{[n/2]}{k} + \binom{[(n+1)/2]}{k}}.$$

Proof It suffices to verify that for each $0 \le i \le n/2$ and l, k subject to conditions of the lemma, we have

$$\frac{B_i}{A_i} \le \frac{\binom{[n/2]}{l} + \binom{[(n+1)/2]}{l}}{\binom{[n/2]}{k} + \binom{[(n+1)/2]}{k}}.$$

This inequality is equivalent to the inequality $\varphi(l,i) < \varphi(k,i)$ from part 3) of Proposition 2.3. $\qquad \square$

Let us turn to the proof of Nikulin's estimate.

Proof of Nikulin's estimate. The estimate of the average number of l-faces on k-faces of a simple n-polytope follows immediately from the theorem on the h-vector. Indeed, firstly, each l-face of a simple n-polytope is contained in exactly $\binom{n-l}{n-k}$ k-dimensional faces of the polytope. Secondly, the number f_m of m-faces of a simple polytope is determined by its h-vector for every m, namely, $f_m = \sum_i \binom{i}{m} h_i$. The desired average number equals to

$$\binom{n-l}{n-k} \frac{\sum_i \binom{i}{l} h_i}{\sum_i \binom{i}{k} h_i}.$$

By the Dehn–Sommerville duality, this number equals to

$$\binom{n-l}{n-k} \frac{\sum\limits_{0 \le i \le n/2} (\binom{i}{l} + \binom{n-i}{l}) \tilde{h}_i}{\sum\limits_{0 \le i \le n/2} (\binom{i}{k} + \binom{n-i}{k}) \tilde{h}_i}, \tag{1}$$

where $\tilde{h}_i = h_i$ for $0 \le i < n/2$ and $\tilde{h}_{[n/2]} = \frac{1}{2} h_{[n/2]}$ for even n. Now apply Lemma 2.1 for $m = 1 + [n/2]$, $A_i = \binom{i-1}{k} + \binom{n-i+1}{k}$, $B_i = \binom{i-1}{l} + \binom{n-i+1}{l}$ and $\alpha_i = \tilde{h}_{i-1}$ (Lemma 2.1 is applicable, since according to part 2) of the theorem on the h-vector, the numbers $h_0, \ldots, h_{[n/2]}$ are nonnegative, and their sum is positive). According to Lemma 2.4, the maximum C of the ratio A_i/B_i is attained for $i - 1 = [n/2]$. According to Lemma 2.1, the value (1) is strictly less than C, since $h_0 > 0$. Nikulin's estimate is proved.   □

Remark 2 The proof of Nikulin's estimate made use of only parts 1) and 2) of the theorem on the h-vector of a simple polytope.

The remark motivates the following plan: try to find a simpler proof for the symmetry of the h-vector and for the non-negativity of its components. This proof should be simpler than a complete proof of the theorem on the h-vector. This would allow to simplify the proof of Nikulin's estimate, and possibly this would allow to generalize it.

Proof of Proposition 1.3 There exists a sequence Δ^N of simple n-dimensional polytopes such that the h-vector components h_i^N of these polytopes have the following property: for any $i < [n/2]$, the limit $\lim_{N \to \infty} h_i^N / h_{[n/2]}^N$ is equal to zero. It suffices to define Δ^N as the polytope dual to an n-dimensional cyclic polytope with N vertices. From formula (1) for the average number of l-faces on k-faces of simple n-polytopes, it follows that the sequence Δ^N provides an example for both parts of Proposition 1.3.   □

3 The theorem on the h-vector and Morse theory

The proof of the necessity of conditions on the h-vector of a simple polytope is based on the theory of Newton polytopes, which relates geometry of polytopes with algebraic geometry of toric varieties.

Let Δ be a convex integer polytope in $\mathbb{R}^n$, i.e. the vertices of the polytope belong to the lattice $\mathbb{Z}^n$. With each integer point $m \in \mathbb{Z}^n$, we can associate the monomial $\chi_m : (\mathbb{C}^*)^n \to \mathbb{C}$ defined by the formula $\chi_m(z_1, \ldots, z_n) = z_1^{m_1} \ldots z_n^{m_n}$.

Denote by A the finite set $A = \Delta \bigcap \mathbb{Z}^n$ of integer points. The *Veronese map* $V_\Delta : (\mathbb{C}^*)^n \to \mathbb{C}P^{N-1}$, where N is the number of points in the set A, is defined as the map taking each point $z \in (\mathbb{C}^*)^n$ to the point with homogeneous coordinates $[\chi_{m_1}(z) : \cdots : \chi_{m_N}(z)]$, where $m_1, \ldots, m_N$ are the points of the set A taken in an arbitrary order (the Veronese map is defined up to a permutation of the set A, however, the property of this map we are interested in does not depend on the choice of the ordering).

The toric variety M_Δ is the normalized projective closure of the image $V_\Delta((\mathbb{C}^*)^n)$ of the group $(\mathbb{C}^*)^n$ under the Veronese map (if the polytope Δ is "not too small", then the projective closure is automatically normal and so it does not need to be normalized). The natural action of the group $(\mathbb{C}^*)^n$ extends to M_Δ. With respect to of this action, M_Δ splits into a finite number of orbits.

If the polytope Δ is simple, then the algebraic variety M_Δ is so called *quasi-smooth variety* (i.e. an orbifold). Quasi-smooth varieties possess many properties of smooth algebraic varieties. In particular, the main results of Hodge theory persist for these varieties. In the sequel, in our heuristic arguments, we will assume that M_Δ is a smooth manifold.

Stanley's proof of the necessary conditions on the h-vector is based on the following fact. It turns out that *the number h_i of a simple integer polytope Δ coincides with the $2i$-th Betti number of the manifold M_Δ*. After this observation, all necessary conditions on the h-vector of a simple integer polytope Δ follow from the theory of toric varieties. Namely: the non-negativity of the numbers h_i becomes obvious, the Dehn–Sommerville duality follows from Poincaré duality $\dim H^i = \dim H^{2n-i}$. The unimodality of the numbers h_i follows from the hard Lefschetz theorem, the inequalities $h_{i+1} - h_i < Q_i(h_i - h_{i-1})$ follow from the fact that the cohomology ring of the manifold M_Δ is generated by the elements of the vector space $H^2(M_\Delta)$. (The function Q_i appears in the Macaulay theorem from commutative algebra (see [16]) describing the Hilbert functions of the quotients of the polynomial ring in several variables.)

The necessary conditions on the h-vector for simple but non-integer polytopes can be easily reduced to the integer case. To perform this reduction, one can do a small perturbation of the facets of the polytope to make them rational. Then all vertices of the polytope become rational as well, and the combinatorial type of the polytope remains unchanged, since the original polytope was simple. After that, we can make all vertices integer by a suitable dilation of the polytope (multiplying by the common denominator of all vertices).

As we have seen, to prove Nikulin's estimate, it suffices to use only the positivity of the numbers h_i and their symmetry $h_i = h_{n-i}$. Positivity of Betti numbers of the manifold M_Δ and Poincaré duality are responsible for these properties. Thus we use neither the existence of the ring structure on the cohomology space of M_Δ, nor the hard Lefschetz theorem.

Morse theory helps frequently to compute Betti numbers. One of the simplest proofs of Poincaré duality is also based on this theory. Hence it is natural to try to use Morse theory for a proof of parts 1) and 2) of the theorem on the h-vector. To this end, we need to consider a simple enough function on M_Δ. To construct such a function, we can use the moment map (see [17]). The moment map $M : M_\Delta \to R^n$ has the following property. First, it takes the manifold M_Δ to the polytope Δ. Second, it establishes a one-to-one correspondence between the orbits of the group $(\mathbb{C}^*)^n$ in M_Δ and the faces of the polytope Δ. Namely, each orbit of (real) dimension $2i$ is mapped by the moment map to the interior of the corresponding i-dimensional face of the polytope.

A linear function on the polytope is said to be *generic*, if on no edge of the polytope does it restrict to a constant.

Definition 4 The *index* of a generic linear function at a vertex of a simple n-dimensional polytope is the number of edges containing this vertex and such that the function decreases along them (in this case, the function increases along the $(n-i)$ remaining edges containing the vertex of index i).

It is easy to verify the following claim.

Proposition 3.1 *Let L be a generic linear function on the polytope Δ. Then the function $L \circ M : M_\Delta \to \mathbb{R}$ on the manifold M_Δ is a Morse function, and its critical points are exactly zero-dimensional orbits of M_Δ. The Morse index of a critical point $A \in M_\Delta$ equals to twice the index of the vertex $M(A)$ of the polytope Δ with respect to the linear function L.*

The connection between the Morse index of the function $L \circ M$ at the point A (a zero-dimensional orbit of M_Δ) and the index of the linear function L at the vertex $M(A)$ of the polytope Δ, admits the following explanation. Let the index of the function L at the vertex $M(A)$ be equal to i. By definition, the vertex $M(A)$ must belong to a face Γ_1 of dimension i and to a face Γ_2 of dimension $(n-i)$ such that the maximum (respectively, minimum) of the function L on Γ_1 (respectively, on Γ_2) is attained at $M(A)$. The pre-images of the faces Γ_1 and Γ_2 under the moment map M are $2i$-dimensional and $2(n-i)$-dimensional submanifolds of M_Δ, respectively, such that the function $L \circ M$ restricted to these submanifolds attains the maximum (respectively, the minimum) at the point A.

The existence of such submanifolds shows that the Morse index of the point A equals to $2i$. Thus the function $L \circ M$ on M_Δ has critical points of even indices only. Hence all odd Betti numbers of the manifold M_Δ are zero, and the number $\dim H^{2i}(M_\Delta)$ is equal to the number of vertices of the polytope Δ, where the function L has index i. However, as we have mentioned before, the number $\dim H^{2i}(M_\Delta)$ equals to h_i. Hence the following theorem must be true, whose statement is absolutely elementary (it involves neither algebraic geometry nor topology).

Theorem 3.2 *For any generic linear function L on a simple polytope Δ, the number $h_i(L)$ of vertices of the polytope Δ, where the index of the function L is i, does not depend on the function L and coincides with the number h_i of the polytope Δ.*

This theorem has a very simple elementary proof, which is given in the next section.

Remark 3 The elementary proof of Theorem 3.2, together with Proposition 3.1, gives the simplest computation of Betti numbers for the manifold M_Δ: all odd Betti numbers are zero, and $\dim H^{2i}(M_\Delta) = h_i$, where h_i is the i-th component of the h-vector of the polytope Δ for $0 \leq i \leq n$.

Remark 4 It turned out that Theorem 3.2 and its elementary proof from the next section had been known (see [18]) before the article [3], and some close arguments were used even earlier (see [19],[20]). However, neither the connection of Theorem 3.2 with the theory of toric varieties and Morse theory, nor the elementary deduction of Nikulin's estimate from Theorem 3.2 (see Corollary 4.3 and Section 5) had been known.

4 Generic linear function on a simple polytope[0]

Let us give an elementary proof of Theorem 3.2 and discuss its geometric corollaries.

Proof of Theorem 3.2 Consider the set of faces of a simple n-dimensional polytope Δ (we mean the set of faces of all dimensions, including vertices, as well as the polytope Δ itself). Let us map this set into the set of vertices of the polytope. To each face, we assign the vertex, where the restriction of L to this face attains

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[0]Linear functions on general convex polytopes, and, in particular, the problem of maximizing such functions, are studied in linear programming. Linear programming has a substantial practical value. Its creator, a distinguished mathematician and economist L.V. Kantorovich (1912–1986), was awarded the Nobel prize in Economics, largely for his classical works on linear programming. In the last years of his life, Leonid Vital'evich was doing Economics, but he preserved a live interest to mathematics. He had read the article [3] and was very enthusiastic about it.

the maximum. It is clear that the pre-image of a vertex A under this map contains exactly $\binom{i}{k}$ faces of dimension k, where i is the index of the function L at the vertex A. Each k-dimensional face belongs to the pre-image of some vertex, hence for each k, $0 \leq k \leq n$, we obtain the equality $f_k = \sum_i \binom{i}{k} h_i(L)$. The collection of all these

equalities is equivalent to the identities $h_i = h_i(L)$, which prove the theorem.

Indeed, the equalities we obtained mean that the polynomial

$$\sum_{0 \leq i \leq n} h_i(L)(t+1)^i$$

coincides with the polynomial

$$\sum_{0 \leq i \leq n} f_i t^i.$$

But, by definition, the polynomial $\sum_{0 \leq i \leq n} h_i(t+1)^i$ also has this property. Hence $h_i = h_i(L)$, as desired. □

Corollary 4.1 *For every n-dimensional simple polytope, all numbers h_i are non-negative for $0 \leq i \leq n$, and the numbers h_0 and h_n are equal to 1.*

Indeed, the number of vertices of index i is nonnegative, and every generic linear function on the polytope has exactly one minimum and exactly one maximum.

Corollary 4.2 *For every n-dimensional simple polytope, we have the Dehn–Sommerville duality, i.e. $h_i = h_{n-i}$.*

Indeed, for any generic linear function L, according to Theorem 3.2, we have $h_i = h_i(L)$. For the computation of the numbers h_{n-i}, we can use the function $-L$. According to Theorem 3.2, we have $h_{n-i} = h_{n-i}(-L)$. ¿From the definition of index we see that the numbers $h_i(L)$ and $h_{n-i}(-L)$ are equal. Corollary 4.2 is thus proved.

Corollary 4.3 *The estimate from Nikulin's theorem holds.*

Indeed, the proof of Nikulin's theorem given in Section 2 uses parts 1) and 2) of the theorem on the h-vector together with the elementary lemma from Section 2. Corollaries 4.1 and 4.2 prove parts 1) and 2) of the theorem on the h-vector. Hence we obtain a simple elementary proof of Nikulin's estimate. In Section 5, we will rewrite this proof separately, without using the notion of the h-vector.

Let us now discuss other corollaries of Theorem 3.2. Corollary 4.1 can be easily strengthened. The following holds:

Corollary 4.4 *Under the assumptions of Corollary 4.1, all the numbers h_i are strictly positive.*

Proof For any fixed vertex of the polytope, we can choose a linear function L so that the index of the function L at this vertex is any given number from 0 to n. But according to the theorem, the numbers $h_i(L)$ do not depend on the choice of L and are equal to h_i, which proves Corollary 4.4. □

Corollary 4.5 *Consider an arbitrary n-dimensional simple polytope and an arbitrary affine hyperplane containing no vertices of the polytope. Under these conditions, there exists a face of the polytope of dimension $[n/2]$ that is disjoint from the hyperplane.*

Proof Perturbing the affine hyperplane slightly, if necessary, we can arrange that it is a level hypersurface $L = c$ of a linear function L generic with respect to the polytope. According to the proof of Corollary 4.4, there exists a vertex A such that the index of the function L at it equals to $[n/2]$. There exists an $[n/2]$-face Γ_1 of the polytope such that the maximum of the function L restricted to this face is attained at the vertex A, and there exists $(n - [n/2])$-dimensional face Γ_2 of the polytope such that the minimum of the function L restricted to this face is attained at the vertex A. If $L(A) < c$, then the hyperplane $L = c$ is disjoint from the face Γ_1; if $L(A) > c$, then the hyperplane $L = c$ is disjoint from the face Γ_2. $\square$

In Sections 7 and 8, we will generalize Corollary 4.5 and give an estimate for the number and the ratio of the faces of different dimensions disjoint from a generic hyperplane section.

5 An elementary proof of Nikulin's estimate

Let us rewrite the proof of Nikulin's inequalities from Corollary 4.3 without using the notion of the h-vector.

Let Δ be a simple n-dimensional polytope, let l and k be integers satisfying the inequalities $0 \leq l < k \leq (n+1)/2$, and let m be the number of vertices of the polytope Δ. Denote by $V_1, \ldots, V_m$ the vertices of this polytope taken in any order.

Fix a generic linear function L on the polytope Δ. To each face of dimension j, where j is any nonnegative integer not exceeding n, assign the vertex where the function L restricted to the face attains its maximum. We obtain the equality

$$f_j = \sum_{1 \leq i \leq m} \binom{\mathrm{ind}(V_i)}{j},$$

where $\mathrm{ind}(V_i)$ is the index of the function L at the vertex V_i. Analogously, to any face, assign the vertex where L attains the minimum. Then we obtain the equality

$$f_j = \sum_{1 \leq i \leq m} \binom{n - \mathrm{ind}(V_i)}{j}.$$

Therefore, we have

$$2f_j = \sum_{0 \leq i \leq m} \binom{\mathrm{ind}(V_i)}{j} + \binom{n - \mathrm{ind}(V_i)}{j}.$$

Let us take into account that each l-dimensional face of a simple n-dimensional polytope is contained exactly in $\binom{n-l}{n-k}$ its k-dimensional faces. We obtain that the average number of l-dimensional faces on k-dimensional faces of the polytope Δ equals to

$$\binom{n-l}{n-k} \frac{\sum\limits_{1 \leq i \leq m} \left(\binom{\mathrm{ind}(V_i)}{l} + \binom{n-\mathrm{ind}(V_i)}{l} \right)}{\sum\limits_{1 \leq i \leq m} \left(\binom{\mathrm{ind}(V_i)}{k} + \binom{n-\mathrm{ind}(V_i)}{k} \right)}.$$

Set

$$A_i = \binom{\mathrm{ind}(V_i)}{k} + \binom{n - \mathrm{ind}(V_i)}{k}, \quad B_i = \binom{\mathrm{ind}(V_i)}{l} + \binom{n - \mathrm{ind}(V_i)}{l}.$$

Using Lemma 2.4 and Corollary 4.4, by which among the vertices of the polytope there is a vertex V_j such that $\text{ind}(V_j) = [n/2]$, we obtain that

$$\max_i \frac{B_i}{A_i} = \frac{\binom{[n/2]}{l} + \binom{[(n+1)/2]}{l}}{\binom{[n/2]}{k} + \binom{[(n+1)/2]}{k}}.$$

Using Lemma 2.1 for the number m being the number of vertices of the polytope Δ, for the numbers A_i and B_i introduced above and for $\alpha_i \equiv 1$, we obtain a proof of Nikulin's theorem. We only need to notice that for $1 < k$, the average number of faces under the estimate is strictly less than $\max_i \frac{B_i}{A_i}$, since among the vertices of the polytope there exist points of maximum and points of minimum of the function L, at which the corresponding ratio is strictly less than the maximal one. The proof of Nikulin's inequalities is completed.

6 Sections of a simplex and a geometric meaning of Nikulin's estimates

In this Section, we will need several simple formulas concerning the combinatorics of hyperplane sections of a simplex. First we present this formulas and show that they play a certain role in Nikulin's estimate. After that, we discuss a plan of a proof of a certain generalization of Nikulin's estimate.

Consider a section of an $(n-1)$-dimensional simplex by an affine hyperplane $L = c$ not passing through its vertices. Suppose that i vertices of the simplex lie on one side of the hyperplane, and $(n-i)$ vertices lie on the other side, where i is any number such that $0 \leq i \leq n$. Then:

1) for $j > 0$, the number f_j^c of $(j-1)$-faces of the simplex disjoint from the hyperplane $L = c$, is equal to

$$\binom{i}{j} + \binom{n-i}{j}.$$

Indeed, on one side of the hyperplane there are $\binom{i}{j}$ such faces, and $\binom{n-i}{j}$ such faces are on the other side;

2) for $0 < k \leq \max(i, (n-i))$, twice the number f_{k-1} of $(k-1)$-faces of the simplex divided by the number f_{k-1}^c satisfies the equality

$$\frac{2f_{k-1}}{f_{k-1}^c} = \binom{n}{k} \frac{2}{\binom{i}{k} + \binom{n-i}{k}};$$

3) for l, k such that $0 < l < k$, the number $f_{l-1,k-1}^c$ of pairs of faces $\Gamma_1 \subset \Gamma_2$ of the simplex, where Γ_1 is a face of dimension $(l-1)$ disjoint from the hyperplane $L = c$, and Γ_2 is a $(k-1)$-face, satisfies the equality

$$f_{l-1,k-1}^c = \binom{n-l}{n-k} \left(\binom{i}{l} + \binom{n-i}{l} \right).$$

Indeed, the number of $(k-1)$-faces of the simplex containing a fixed $(l-1)$-face, is $\binom{n-l}{n-k}$. It now remains to use the equality from 1);

4) for l,k such that $0 < l < k \leq \max(i, (n-i))$, the number $f_{l-1,k-1}^c$ divided by the number f_{k-1}^c satisfies the equality

$$\frac{f_{l-1,k-1}^c}{f_{k-1}^c} = \binom{n-l}{n-k} \frac{\binom{i}{l} + \binom{n-i}{l}}{\binom{i}{k} + \binom{n-i}{k}}.$$

Proposition 6.1 *a) For $0 < k \leq (n+1)/2$, the maximal value of the ratio $\frac{2f_{k-1}}{f_{k-1}^c}$ for a generic section $L = c$ of the simplex equals to*

$$\frac{2\binom{n}{k}}{\binom{[n/2]}{k} + \binom{[(n+1)/2]}{k}}.$$

b) For $0 < l < k \leq (n+1)/2$, the maximal value of the ratio $\frac{f_{l-1,k-1}^c}{f_{k-1}^c}$ for a generic section $L = c$ of the simplex equals to

$$\binom{n-l}{n-k} \frac{\binom{[n/2]}{l} + \binom{[(n+1)/2]}{l}}{\binom{[n/2]}{k} + \binom{[(n+1)/2]}{k}}.$$

Indeed, this is readily seen from formulas 2) and 4) for these ratios and from Lemma 2.4.

Theorem 6.2 (on a geometric meaning of Nikulin's estimate) *For l and k such that $0 \leq l < k \leq (n+1)/2$ and $1 < k$, the average number of l-dimensional faces on k-dimensional faces of a simple n-dimensional polytope Δ is strictly less than the maximum over all $(n-1)$-dimensional sections of a simplex by a generic hyperplane $L = c$:*
1) for $l = 0$: of the ratio $\frac{2f_{k-1}}{f_{k-1}^c}$;
2) for $0 < l$: of the ratio $\frac{f_{l-1,k-1}^c}{f_{k-1}^c}$.

Proof This theorem follows from Nikulin's estimate and Proposition 6.1. □

Theorem 6.2 can be proved directly, by translating the proof of Nikulin's inequalities from Section 5 to the language of sections of a simplex. The reason we can perform this translation is the following. Near each vertex, a simple n-dimensional polytope looks like a cone over an $(n-1)$-dimensional simplex. A level hypersurface of a linear function L passing through a vertex of the polytope, gives rise to a section of this $(n-1)$-dimensional simplex. If the index of the function L at the vertex is i, then i vertices of the simplex lie on one side of this section, and $(n-i)$ vertices lie on the other side. This observation allows to perform the desired translation.

Our further plan is as follows. In Section 9, we consider sections of simple $(n-1)$-dimensional polytopes by generic hyperplanes and solve the same problems for them as those we solved in Proposition 6.1 for a simplex. Then, in Section 10, we prove generalized Nikulin's estimates for n-dimensional polytopes simple at the edges. We will use the fact that a polytope simple at the edges looks like a cone over some simple $(n-1)$-dimensional polytope near every its vertex.

We now proceed to the realization of this plan.

7 An estimate for the number of faces of a section

To estimate the average number of l-dimensional faces on k-dimensional faces of an n-dimensional polytope, we need to deal with $(n-1)$-dimensional polytopes, with their $(l-1)$-dimensional and $(k-1)$-dimensional faces and with sections of these polytopes. To avoid the persisting "-1" in dimensions of polytopes and their faces throughout the remaining part of the text, and since the problems on sections of simple polytopes are interesting on their own right, we change the notation

for dimensions. We will speak of s-dimensional faces on r-dimensional faces of q-dimensional polytopes and of hyperplane sections of these q-dimensional polytopes.

Thus let $\Delta \subset \mathbb{R}^q$ be a simple q-dimensional polytope, and let a hyperplane not passing through the vertices of the polytope Δ be fixed. Perturbing the hyperplane slightly, we can arrange that it will be a level hypersurface $L = c$ of a generic linear function L on the polytope Δ. Let O and Π be the sets of vertices of the polytope Δ where the function L is less than c and greater than c, respectively. The set of all vertices of the polytope is the union of the subsets O and Π, since the hyperplane does not pass through the vertices of the polytope.

Theorem 7.1 *The number f_j^c of j-dimensional faces of the polytope Δ disjoint from the hyperplane $L = c$, is given by the formula*

$$f_j^c = \sum_{b \in O} \binom{\mathrm{ind}(b)}{j} + \sum_{b \in \Pi} \binom{q - \mathrm{ind}(b)}{j}. \tag{2}$$

Proof The set of j-dimensional faces disjoint from the hyperplane $L = c$, splits into two subsets: the subset of faces where the function L is strictly greater than c, and the subset of faces where the function L is strictly less than c.

The number of faces in the first set equals to the first summand in formula (2) for f_j^c. To prove this, we associate each face from this set with the vertex, where the restriction of the function L to the face attains the maximum (an analogous argument was used in the proof of Theorem 3.2). Associating the faces from the second set with the minimum points of the function L, we see that the number of faces in the second set is equal to the second summand in formula (2). Theorem is thus proved. $\qquad\square$

We will need formula (2) in next sections for the proof of generalized Nikulin's estimate.

Let us discuss here one interesting corollary of this formula, which is not relevant for this generalization. The corollary allows to give upper bounds for all components of the h-vector of the section of Δ by a generic affine plane of codimension l in terms of the h-vector of the polytope Δ. This, in turn, allows to estimate the numbers of faces of all dimensions of any (not necessarily generic) affine section of the polytope Δ.

To describe this estimate, we will need the following operation S, taking each reciprocal polynomial with nonnegative coefficients to a reciprocal polynomial with nonnegative coefficients, whose degree is one less. By definition, the polynomial $S \circ p_m(t)$ can be constructed from a polynomial $p_m(t)$ of degree m in the following way: the polynomial $S \circ p_m(t)$ is the unique reciprocal polynomial of degree $(m-1)$ such that for $k \geq (m-1)/2$, its coefficient with the monomial t^k coincides with the coefficient with the monomial t^k in the Laurent series for the rational function $p_m(t)(t-1)^{-1}$ at ∞.

Theorem 7.2 *All coefficients of the h-polynomial of any generic hyperplane section of a simple polytope Δ do not exceed the corresponding coefficients of the polynomial $S \circ h(t)$, where $h(t)$ is the h-polynomial of the polytope Δ.*

Proof Denote by $f_j^{<c}$ and by $f_j^{>c}$ the number of j-dimensional faces of the polytope Δ, lying beneath and, respectively, above the level hypersurface $L = c$. Denote by $h_j^{<c}$ and by $h_j^{>c}$ the number of vertices of index j with respect to the function L lying beneath the level hypersurface $L = c$, and, respectively, the number

of vertices of index $(q - j)$ lying above this hypersurface. Denote by $\tilde{f}_j$ and $\tilde{h}_j$ the j-th components of the f-vector and the h-vector, respectively, of a section of the polytope Δ.

For each j, we have the obvious relation $f_j = f_j^{<c} + \tilde{f}_{j-1} + f_j^{>c}$. Using these relations, we can rewrite formula (2) in the form

$$h(t) = h^{<c}(t) + \tilde{h}(t)(t - 1) + h^{>c}(t),$$

where $h(t), \tilde{h}(t)$ are the h-polynomials of the polytope Δ and of its section, respectively, and $h^{<c}(t), h^{>c}(t)$ are generating polynomials for the sequences $h_j^{<c}$ and $h_j^{>c}$.

Identity (2) means that

$$\tilde{h}(t) = (h(t) - h^{<c}(t) - h^{>c}(t))(t - 1)^{-1}. \tag{3}$$

All coefficients $h_j^{<c}, h_j^{>c}$ of the polynomials $h^{<c}(t), h^{>c}(t)$ are nonnegative. Near the point ∞, we have $(t - 1)^{-1} = t^{-1} + t^{-2} + \ldots$. Hence identity (3) implies Theorem 7.2. $\qquad\square$

Definition 5 A section of a simple q-dimensional polytope by a generic hyperplane is said to be *successful*, if it intersects all faces of dimension $> q/2$ (or, in other words, if it intersects all faces of codimension $\leq (q - 1)/2$).

From formula (2) it is readily seen that a section $L = c$ is successful if and only if at all vertices of index $< q/2$, the values of the function L are less than c, and at all vertices of index $> q/2$ the values are greater than c.

Proposition 7.3 *The upper bounds for the h-vector components of a generic hyperplane section of a simple q-dimensional polytope from Theorem 7.2 are simultaneously attained if and only if the section is successful.*

Indeed, according to formula (3), for all estimates to be simultaneously attained it is necessary that the polynomials $h^{<c}(t)$ and $h^{>c}(t)$ have degree $\leq q/2$. This happens if and only if the section $L = c$ is successful.

Definition 6 A section of a simple q-dimensional polytope by a generic affine plane of dimension l is called *successful*, if it intersects all codimension $\leq l/2$ faces of the polytope.

Theorem 7.4 *For any generic l-dimensional affine section of a simple polytope Δ, all coefficients of the h-polynomial of this section do not exceed the corresponding coefficients of the polynomial $S^{(q-l)} \circ h(t)$, where $h(t)$ is the h-polynomial of the polytope Δ and $S^{(q-l)}$ is the $(q - l)$-th iteration of the operation S. The upper bounds for the h-vector components of the section are simultaneously attained if and only if the section is successful.*

Proof The inequalities from Theorem 7.4 follow from Theorem 7.2, since any section of dimension l can be obtained by taking a hyperplane section, then a hyperplane section of this section, and so on.

The case when all inequalities turn simultaneously into equalities can be worked out in the same way as in Proposition 7.3. $\qquad\square$

From Theorem 7.4 we see how to estimate the f-vector of any generic l-dimensional section of a simple q-dimensional polytope Δ in terms of the f-vector of the polytope.

Let us construct the f-vector of some abstract simple l-dimensional polytope, such that the number $\tilde{f}_{(l-j)}$ of its faces of dimension $(l-j)$ is equal to $f_{(q-j)}$ for $j \leq l/2$. The numbers of faces of smaller dimensions of such a polytope can be recovered using the Dehn–Sommerville duality (from the theorem on the h-vector it is readily seen that there exist simple polytopes with such f-vector.)

Corollary 7.5 *For any k, the number of k-faces of a section of the polytope Δ by a generic affine plane of dimension l does not exceed the component $\tilde{f}_k$ of the f-vector thus constructed.*

Corollary 7.6 *The estimate from Corollary 7.5 holds for a section of the polytope Δ by an arbitrary affine plane of dimension l, which can even be non-generic for the polytope Δ.*

Proof Let a simple polytope Δ be given by linear inequalities $0 \leq L_i$, and suppose that we study a section of the polytope Δ by an affine plane P of dimension l. Consider the one-parameter family of polytopes $\Delta(u)$, given by the inequalities $0 \leq L_{i,u} = L_i + \epsilon_i(u)$, where $\epsilon_i(u)$ are generic linear functions of u that are positive for $u > 0$.

For small positive values of u, all polytopes $\Delta(u)$ are combinatorially equivalent and have the same h-vector. If the functions $\epsilon_i(u)$ are generic, then the polytopes lying in the plane P of codimension l and given there by the inequalities $0 \leq L_{i,u}$ with small positive u are simple. We can apply Corollary 7.5 to those polytopes. Polytopes $P \cap \Delta(u)$ corresponding to different small $u > 0$ have parallel facets, and they give rise to the same partition $\Delta^*(u)$ of the dual space P^*. The polytopes $\Delta(u)$ degenerate to the section $\Delta \cap P$ for $u = 0$.

It is clear that for such degeneration, the number of faces in each dimension does not increase (the partition $\Delta^*(u)$ of P^* dual to the polytope $\Delta(u)$, is a subdivision of the partition Δ_0^*). The corollary is thus proved. $\qquad\square$

Problem 7.7 Let the h-vector of a simple q-dimensional polytope be given. What can be the h-vector of a section of the polytope by a generic affine plane of dimension l?

Theorem 7.4 gives an upper estimate for the components of the h-vector of the section. I think that this estimate is sharp but I can not prove this. For a proof, we need to construct a simple q-dimensional polytope with a given h-vector, such that there exists a generic affine plane of dimension l, intersecting all faces of the polytope of codimension $\leq l/2$.

Remark 5 According to the famous Upper Bound Conjecture, the l-dimensional polytope dual to a cyclic polytope with N vertices has the maximal number of faces in any dimension among all simple l-dimensional polytopes having N facets. This conjecture is proved (see [18–21]). The estimate from the Upper Bound Theorem follows from the partial case of Corollary 7.6 when the polytope Δ is an $(N - 1)$-dimensional simplex. The arguments from Section 7 are very close to the arguments that were used to prove the Upper Bound Conjecture.

8 The ratio of faces disjoint from a section

Let us return to the realization of our plan (see the end of Section 6). Let the h-vector of some simple q-dimensional polytope be fixed. Consider an arbitrary

simple polytope Δ with the given h-vector and fix an arbitrary generic hyperplane section $L = c$ of this polytope.

Denote by f_j, f_j^c the number of j-dimensional faces of the polytope Δ and, respectively, the number of j-dimensional faces of the polytope Δ disjoint from the hyperplane $L = c$. We are interested in the following

Problem 8.1 1) Give an upper estimate for the ratio f_r/f_r^c with any r, $1 \leq r \leq q/2$, in terms of the h-vector.

2) Give an upper estimate for the ratio f_s^c/f_r^c with any s, r, $0 \leq s < r \leq q/2$ in terms of the h-vector.

To state the results on Problem 8.1, let us introduce the following notation. For any vector $h = (h_0, \ldots, h_q)$ with positive (not necessarily integer) components h_i and with the symmetry property $h_i = h_{q-i}$, set:

1) $F_j(h) = \sum\limits_{0 \leq i \leq q} h_i \binom{i}{j}$;

2) $\Phi_j(h) = \sum\limits_{0 \leq i < q/2} 2h_i \binom{i}{j} + Q_j$, where $Q_j = h_{q/2} \binom{q/2}{j}$ for even q and $Q_j = 0$ for odd q.

Theorem 8.2 *For every generic hyperplane section $L = c$ of a simple q-dimensional polytope Δ with the h-vector h, the following inequalities hold:*

1) For any r such that $1 \leq r \leq q/2$,

$$\frac{f_r}{f_r^c} \leq \frac{F_r(h)}{\Phi_r(h)};$$

2) For any s, r such that $0 \leq s < r \leq q/2$,

$$\frac{f_s^c}{f_r^c} \leq \frac{\Phi_s(h)}{\Phi_r(h)}.$$

For a successful section $L = c$ of the polytope Δ, all these inequalities are equalities.

Conversely, if for at least one r satisfying the conditions of part 1), or for at least one pair s, r satisfying the conditions of part 2), the inequality turns to equality, then the section $L = c$ of the polytope Δ is successful.

Remark 6 For any generic hyperplane section of the polytope Δ the ratio $\frac{f_0}{f_0^c}$ is equal to one, since a generic hyperplane intersects no vertices of the polytope Δ.

The proof of Theorem 8.2 is based on the solution of Problem 8.3, which is posed below. With a vector $h = (h_0, \ldots, h_q)$ having positive integer components h_i and having the symmetry property $h_i = h_{q-i}$, associate the collection of sets $V_0, \ldots, V_q$ containing, respectively, $h_0, \ldots, h_q$ elements. The number of elements in a finite set A will be denoted by $\aleph(A)$.

Problem 8.3 Find partitions $V_i = O_i \cup \Pi_i$ of sets V_i such that

1) for given r, $1 \leq r \leq q/2$, the ratio

$$\frac{\sum\limits_{0 \leq i \leq q} h_i \binom{i}{r}}{\sum\limits_{0 \leq i \leq q} \aleph(O_i) \binom{i}{r} + \aleph(\Pi_i) \binom{q-i}{r}}$$

is maximal;

2) for given s, r, $0 \leq s < r \leq q/2$, the ratio

$$\frac{\sum\limits_{0 \leq i \leq q} \aleph(O_i)\binom{i}{s} + \aleph(\Pi_i)\binom{q-i}{s}}{\sum\limits_{0 \leq i \leq q} \aleph(O_i)\binom{i}{r} + \aleph(\Pi_i)\binom{q-i}{r}} \tag{4}$$

is maximal.

Remark 7 The question from part 1) of Problem 8.3 can be posed even for $r = 0$. But in this case the ratio does not depend on the choice of a partition and is identically equal to 1.

A collection of partitions $V_i = O_i \cup \Pi_i$ of sets V_i is called *successful*, if:
1) for $i < q/2$, the sets O_i and V_i coincide, and the set Π_i is empty,
2) for $i > q/2$, the set O_i is empty, and the sets Π_i and V_i coincide,
3) for $i = q/2$, the partition of the set V_i into subsets O_i and Π_i is arbitrary.
The following theorem provides a complete solution of Problem 8.3.

Theorem 8.4 *1) A successful collection of partitions of the sets $V_0, \ldots, V_m$ maximizes the ratio from part 1) for any r. The desired maximum is $\frac{F_r(h)}{\Phi_r(h)}$.*

2) A successful collection of partitions of the sets $V_0, \ldots, V_m$ maximizes the ratio from part 2) for any s and r. The desired maximum is $\frac{\Phi_s(h)}{\Phi_r(h)}$.

3) If partitions $V_i = O_i \cup \Pi_i$ of the sets V_i maximize the ratio from part 1) for some r or maximize the ratio from part 2) for some s and r, then this collection of partitions is successful.

Let us deduce Theorem 8.2 from Theorem 8.4. A generic linear function L defines a partition of the set V of vertices of the polytope Δ into subsets $V_0, \ldots, V_q$: the subset V_i contains h_i elements and is defined as the set of vertices having index i with respect to the function L. By fixing a level $L = c$, we partition each set V_i into subsets O_i and Π_i, consisting of vertices where $L < c$ and, respectively, $L > c$.

To deduce Theorem 8.2 from Theorem 8.4, it now suffices to compare formula (2) from Theorem 7.1 with Problem 8.3.

First part of Problem 8.3 is very simple. The following lemma is obvious:

Lemma 8.5 *The following value*

$$\sum_{0 \leq i \leq q} \aleph(O_i)\binom{i}{r} + \aleph(\Pi_i)\binom{q-i}{r}$$

attains the minimum on successful partitions and only on them.

According to Lemma 8.5, successful partitions and only them give a solution for part 1) of Problem 8.3.

For the proof of remaining parts of Theorem 8.4 it is convenient to use the *fractional linear programming*. Fractional linear programming is the maximizing of the ratio L_1/L_2 of two linear functions on a convex polytope (it is assumed that the function L_2 vanishes nowhere on the polytope). Fractional linear programming is not very different from linear programming. Indeed, consider an arbitrary projective transformation of the space $\mathbb{R}P^q \supset \mathbb{R}^q$ mapping the hyperplane $L_2 = 0$ to the hyperplane at infinity. The convex polytope gets transformed into another convex polytope, and the fractional linear function L_1/L_2 becomes linear. Thus the original problem transforms to a problem of linear programming.

Hence the set of points where the maximum of a fractional linear function is attained, is a face of the polytope. (In particular, the maximum is attained at a vertex of the polytope. Lemma 2.1 from Section 2 is based on this fact.) In the case of general position, this face is necessarily a vertex of the polytope.

Let us formulate a continuous variant of part 2) of Problem 8.3. Let $h = (h_0, \ldots, h_q)$ be a vector with positive (not necessarily integer) components h_i, and with the symmetry property $h_i = h_{q-i}$. Consider the parallelepiped Δ in the space $\mathbb{R}^{[(q+1)/2]}$, defined by the inequalities $0 \leq x_i \leq 2h_i$ for $0 \leq i < q/2$ (the number $[(q+1)/2]$ is the number of indices i satisfying the inequalities $0 \leq i < q/2$). For each integer j, define a linear function on $\mathbb{R}^{[(q+1)/2]}$ by the formula

$$L_j = \sum_{0 \leq i < q/2} (x_i \binom{i}{j} + (2h_i - x_i)\binom{q-i}{j}) + Q_j,$$

where $Q_j = h_{q/2}\binom{q/2}{j}$ for even q and $Q_j = 0$ for odd q;

Consider the following problem of fractional linear programming.

Problem 8.6 Maximize the function L_s/L_r on the parallelepiped Δ, where s, r are fixed numbers satisfying the inequalities $0 \leq s < r \leq q/2$.

Theorem 8.7 *For any s, r satisfying the conditions of Problem 8.6, the strict maximum of the function L_s/L_r is attained at the vertex Γ of the parallelepiped such that its i-th coordinate x_i is $2h_i$ for $0 \leq i < q/2$. This maximum $L_s(\Gamma)/L_r(\Gamma)$ is equal to $\Phi_s(h)/\Phi_r(h)$.*

Theorem 8.7 allows to conclude the proof of Theorem 8.4 started in Lemma 8.3. For a fixed collection of partitions $V_0 = O_0 \cup \Pi_0, \ldots, V_q = O_q \cup \Pi_q$ and for each $0 \leq i < q/2$, set $x_i = \chi(O_i) + \chi(\Pi_{q-i})$. Then the value $L_s(x)/L_r(x)$ at the point x, where $x \in \mathbb{R}^{[(q+1)/2]}$ is a vector with coordinates x_i, equals to the value of ratio (4) for the given collection of partitions of the sets V_i.

Furthermore, $x_i = 2h_i$ if and only if $\chi(O_i) = h_i$ and $\chi(\Pi_{q-i}) = h_i$. Hence, after we prove Theorem 8.7, Theorem 8.4 will be proved completely.

The proof of Theorem 8.7 uses the following property of binomial coefficients.

Lemma 8.8 *Suppose that $s < r$. Then the ratio $\psi(m) = \binom{m}{s}/\binom{m}{r}$ strictly decreases as m runs from r to ∞, and the ratio $\varphi(m) = \binom{r}{m}/\binom{s}{m}$ strictly increases as m runs from 0 to s.*

Indeed, the denominator of the ratio

$$\psi(m) = (r!/s!)(1/(m-s)(m-s-1)\ldots(m-r+1))$$

increases as m increases, and the numerator of the ratio

$$\varphi(m) = (r!/s)!(s-m)\ldots(r-m+1)$$

increases as m increases.

Proof of Theorem 8.7.

Step 1. The value $L_s(\Gamma)/L_r(\Gamma)$ at the vertex Γ is bigger than $\binom{[q/2]}{s}/\binom{[q/2]}{r}$. Indeed, $L_s(\Gamma)/L_r(\Gamma) > L_{s,r}(\Gamma)/L_r(\Gamma)$, where

$$L_{s,r}(\Gamma) = \sum_{r \leq i < q/2} 2h_i \binom{i}{s} + Q_s.$$

According to Lemma 8.8 applied to the function $\psi(m)$, the numbers $\binom{i}{s}/\binom{i}{r}$ increase as i increases from $i = r$. To complete step 1, it remains to use Lemma 2.1 (it is applicable, since the numbers h_i are strictly positive for $r \leq i \leq [n/2]$).

Step 2. The value $L_s(V)/L_r(V)$ at any vertex V adjacent to the vertex Γ (i.e. at a vertex V that is connected with Γ by an edge), is strictly less than the corresponding value at the vertex Γ. Indeed, all coordinates of the vertex V, except only one, coincide with coordinates of the vertex Γ. Let the index of this special coordinate be i. If the vertex Γ gets replaced with the vertex V, then the value of the function L_s increases by $2h_iB$, where $B = \binom{q-i}{s} - \binom{i}{s}$, and the value of the function L_r increases by the number $2h_iA$, where $A = \binom{q-i}{r} - \binom{i}{r}$. By Step 1, the number $\binom{[q/2]}{s}/\binom{[q/2]}{r}$ is at most $L_s(\Gamma)/L_r(\Gamma)$.

Let us show that $B/A < \binom{[q/2]}{s}/\binom{[q/2]}{r}$. The inequality $B/\binom{[q/2]}{s} < A/\binom{[q/2]}{r}$ follows from Lemma 8.8 applied to the function $\varphi(m)$. Indeed, $\binom{q-i}{s}/\binom{[q/2]}{s} < \binom{q-i}{r}/\binom{[q/2]}{r}$, since $(q-i) > [q/2]$ and $s < r$. Furthermore,

$$-\frac{\binom{i}{s}}{\binom{[q/2]}{s}} \leq -\frac{\binom{i}{r}}{\binom{[q/2]}{r}}$$

(For $i < r$ and for $i = [q/2]$, this inequality turns to equality. For $r \leq i < [q/2]$ and $s < r$, this inequality is strict and is equivalent to the relation $\binom{[q/2]}{s}/\binom{i}{s} < \binom{[q/2]}{r}/\binom{i}{r}$, which also follows from Lemma 8.8 applied to the function $\varphi(m)$.)

Summing up the two obtained inequalities, we arrive at the desired result.

Step 3. Fractional linear programming and the result of Step 2 prove that the function L_s/L_r attains its maximum at the vertex Γ. Theorem 8.7, together with Theorem 8.4 and 8.2, is proved. $\qquad\square$

9 Extremal property of sections of a simplex

In the statement of Theorem 9.1, which is the central result of this section, we use the notation introduced in Section 8.

Theorem 9.1 *For every generic hyperplane section $L = c$ of a simple q-dimensional polytope Δ, the following inequalities hold, which turn to equalities for a successful section of a q-dimensional simplex:*

1) For $1 \leq r \leq q/2$,

$$\frac{f_r}{f_r^c} \leq \frac{\binom{q+1}{r+1}}{\binom{[(q+1)/2]}{r+1} + \binom{[(q+2)/2]}{r+1}};$$

2) For $0 \leq s < r \leq q/2$,

$$\frac{f_s^c}{f_r^c} \leq \frac{\binom{[(q+1)/2]}{s+1} + \binom{[(q+2)/2]}{s+1}}{\binom{[(q+1)/2]}{r+1} + \binom{[(q+2)/2]}{r+1}}.$$

If for at least one r from the inequality of part 1) we have the equality, then the polytope Δ is a simplex, and its section $L = c$ is successful. If for some pair s, r the inequality from part 2) is the equality, then the components h_i of the h-vector of the polytope Δ are equal to each other for $s \leq i \leq (q-s)$, and the section $L = c$ of the polytope Δ is successful.

In the proof of Theorem 9.1, we will use the fact that the h-vector of a simple polytope is unimodal (see Section 3). This is a part of the theorem on the h-vector.

Remark 8 For some simple polytopes, the unimodality of the h-vector is obvious, and does not require the use of the theorem on the h-vector. For example, the h-vector of a direct product of simplices possesses this property, since the h-polynomial of the direct product of polytopes is the product of their h-polynomials. This fact, together with the theorem on reduction from Section 10, proves Nikulin's estimate for almost simple polytopes (i.e. it proves Corollary 0.4) without using the theorem on the h-vector.

We will need Problem 9.2 posed below. A solution to this problem is given by Theorem 9.3, which implies Theorem 9.1 immediately. Consider the set of vectors $h = (h_0, \ldots, h_q)$ such that the components h_i of these vectors are:

1) positive,

2) symmetric, i.e. $h_i = h_{q-i}$,

3) unimodal, i.e. $0 \leq h_0 \leq \cdots \leq h_{[q/2]}$.

By the symmetry condition, a vector h is determined by its components $h_0, \ldots, h_{[q/2]}$, the number of which is $[q/2] + 1$. Consider a simplex Δ in $\mathbb{R}^{[q/2]+1}$ defined by the inequalities $0 \leq h_0 \leq \cdots \leq h_{[q/2]}$ and the equation $h_0 + \cdots + h_{[q/2]} = v$, where v is an arbitrary positive number.

Problem 9.2 Maximize on the simplex Δ:

1) for $1 \leq r \leq q/2$, the function $\frac{F_r(h)}{\Phi_r(h)}$,

2) for $0 \leq s < r \leq q/2$, the function $\frac{\Phi_s(h)}{\Phi_r(h)}$.

First note that the maximum in Problem 9.2 does not depend on the choice of constant v, since the functions we maximize are homogeneous of degree 0. A complete answer to this problem is given by the following

Theorem 9.3 *1) For $1 \leq r \leq q/2$, the maximum of the function $\frac{F_r(h)}{\Phi_r(h)}$ is attained for $h_0 > 0, h_0 = \cdots = h_{[q/2]}$, and is equal to*

$$\frac{\binom{q+1}{r+1}}{\binom{[(q+1)/2]}{r+1} + \binom{[(q+2)/2]}{r+1}}.$$

2) For $0 \leq s < r \leq q/2$, the maximum of the function $\frac{\Phi_s(h)}{\Phi_r(h)}$ is attained for $h_0 + \cdots + h_s > 0, h_s = \cdots = h_{[q/2]}$, and is equal to

$$\frac{\binom{[(q+1)/2]}{s+1} + \binom{[(q+2)/2]}{s+1}}{\binom{[(q+1)/2]}{r+1} + \binom{[(q+2)/2]}{r+1}}.$$

Let us deduce Theorem 9.1 from Theorem 9.3. According to Theorem 8.2, for a fixed h-vector h of the polytope Δ, the ratios $\frac{f_r}{f_r^c}$ and $\frac{f_s^c}{f_r^c}$ do not exceed the ratios $\frac{F_r(h)}{\Phi_r(h)}$ and $\frac{\Phi_s(h)}{\Phi_r(h)}$, and the equality is attained for successful sections only. To conclude the deduction of Theorem 9.1 from Theorem 9.3, it remains to note that a q-dimensional simplex is the only simple polytope such that all components of its h-vector are equal. Indeed, since $h_0 = 1$, all components of the h-vector must be equal to 1. It follows that $f_q = q + 1$, i.e. that the polytope is a simplex.

For the proof of Theorem 9.3, we will need simple Lemmas 9.4 and 9.5 on sums of binomial coefficients. We will also use classical Abel's lemma, which is a discrete variant of the integration by parts, together with its application to the functions we maximize (Lemma 9.6). Finally, we will need a simple general fact from fractional

linear programming (Lemma 9.7) and two simple lemmas dealing with the functions we maximize (Lemmas 9.8 and 9.9).

Lemma 9.4 (on a sum of binomial coefficients) *The following formula holds:*

$$\sum_{i \leq j \leq k} \binom{j}{m} = \binom{k+1}{m+1} - \binom{k+1}{j+1}.$$

Proof Computing the sum of a geometric series, we obtain the identity

$$\sum_{i \leq j \leq k} (1+t)^j = ((1+t)^{k+1} - (1+t)^{i+1})/t.$$

Equating the coefficients with t^m in this identity, we obtain the required equality.
□

Lemma 9.5 *The following formula holds:*

$$\sum_{i \leq k \leq (q-i)} \binom{\min(k, q-k)}{r} = \binom{[(q+1)/2]}{r+1} + \binom{[(q+2)/2]}{r+1} - 2\binom{i+1}{r+1}.$$

Proof The desired sum can be rewritten in the form

$$\sum_{i \leq k \leq [(q-1)/2]} \binom{k}{r} + \sum_{i \leq k \leq [q/2]} \binom{k}{r}.$$

Applying the previous lemma to each sum, we obtain the required equality.

To make the use of the unimodality condition for the h-vector simpler, let us transform the functions we maximize. Recall the following discrete variant of the integration by parts. For every sequence $a_0, \ldots, a_n$, define sequences $(\Delta a)_i$ and $(Sa)_i$, where $(\Delta a)_i = a_i - a_{i-1}$ for $0 < i \leq n$, and $(\Delta a)_0 = a_0$, and $(Sa)_i^n = \sum_{i \leq j \leq n} a_j$ for $0 \leq i \leq n$.
□

Abel's lemma *For any pair of sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$, the following equality holds:*

$$\sum_{0 \leq i \leq n} a_i b_i = \sum_{0 \leq i \leq n} (\Delta a)_i (Sb)_i^n.$$

Lemma 9.6 *We have the following equalities:*
1) $F_j(h) = \sum_{0 \leq i \leq [q/2]} (\Delta h)_i \sum_{i \leq k \leq q-i} \binom{k}{j};$
2) $\Phi_j(h) = \sum_{0 \leq i \leq [q/2]} (\Delta h)_i \sum_{i \leq k \leq (q-i)} \binom{\min(k, q-k)}{j}.$

This follows from the Abel lemma for $n = [q/2]$ and the sequence $a_i = h_i$. The sequence b_i for the function $F_j(h)$ is defined by the following relations: $b_i = \binom{i}{j} + \binom{q-i}{j}$ for $i < q/2$, and $b_{q/2} = \binom{q/2}{j}$ for even q, and for the function $\Phi(h)$ by the relations: $b_i = 2\binom{i}{j}$ for $i < q/2$, and $b_{q/2} = \binom{q/2}{j}$ for even q.

Lemma 9.7 *Let all components of the vectors $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$ be strictly positive and $\frac{B_1}{A_1} > \cdots > \frac{B_n}{A_n}$. Then the numbers*

$$D_i = \frac{\sum_{i \leq j \leq n} B_j}{\sum_{i \leq j \leq n} A_j}$$

satisfy the inequalities $\frac{B_1}{A_1} > D_1 > \cdots > D_n = \frac{B_n}{A_n}$.

Proof Suppose that for some $i > 1$ we proved that $\frac{B_i}{A_i} \geq D_i$. Then, by Lemma 2.1,

$$D_{i-1} = \frac{B_{i-1} + (\sum_{i \leq j \leq n} B_i)}{A_{i-1} + (\sum_{i \leq j \leq n} A_i)}$$

is strictly bigger than D_i *and strictly smaller than* $\frac{B_{i-1}}{A_{i-1}}$. The lemma is thus proved. $\square$

Lemma 9.8 *1) For* $1 \leq r \leq q/2$, *the numbers*

$$D_i = \frac{\sum\limits_{i \leq k \leq q-i} \binom{k}{r}}{\sum\limits_{i \leq k \leq (q-i)} \binom{\min(k,q-k)}{r}}$$

strictly decrease as i *runs from 0 to* $[q/2]$.

2) We have

$$\max_{0 \leq i \leq [q/2]} D_i = D_0 = \frac{\binom{q+1}{r+1}}{\binom{[(q+1)/2]}{r+1} + \binom{[(q+2)/2]}{r+1}}. \tag{5}$$

Proof 1) The numbers $\frac{\binom{i}{r} + \binom{n-i}{r}}{2\binom{i}{r}}$ strictly decrease as i increases on the semi-segment $r \leq i < q/2$. Moreover, for even q, all these numbers are bigger than $\frac{\binom{q/2}{r}}{\binom{q/2}{r}} = 1$. It now remains to use Lemma 9.7.

2) The numbers D_i strictly decrease as i increases on the segment $0 \leq i \leq r$. Indeed, as i increases, the numerator

$$\sum_{i \leq k \leq q-i} \binom{k}{r} = \sum_{r \leq k \leq q-i} \binom{k}{r}$$

strictly decreases, whereas the denominator

$$\sum_{i \leq k \leq (q-i)} \binom{\min(k,q-k)}{r} = \sum_{r \leq k \leq (q-r)} \binom{\min(k,q-k)}{r}$$

remains unchanged. Formula (5) for the number D_0 follows from Lemmas 9.4 and 9.5. The lemma is thus proved. $\square$

Lemma 9.9 *For* $0 \leq s < r \leq q/2$,

1) the numbers

$$D_i = \frac{\sum\limits_{i \leq k \leq (q-i)} \binom{\min(k,q-k)}{s}}{\sum\limits_{i \leq k \leq (q-i)} \binom{\min(k,q-k)}{r}}$$

strictly decrease as i *runs from* s *to* $[q/2]$,

2) we have

$$\max_{0 \leq i \leq [q/2]} D_i = D_0 = \cdots = D_s = \frac{\binom{[(q+1)/2]}{s+1} + \binom{[(q+2)/2]}{s+1}}{\binom{[(q+1)/2]}{r+1} + \binom{[(q+2)/2]}{r+1}}. \tag{6}$$

Proof 1) For $r \leq j < q/2$, the number $\binom{j}{s}$ enters the numerator exactly twice: for $k = j$ and for $k = q - j$. Analogously, the number $\binom{j}{r}$ enters the denominator exactly twice. If q is even, then $\binom{q/2}{s}$ and, respectively, $\binom{q/2}{r}$ appear once in the numerator and, respectively, in the denominator.

The numbers $\frac{2\binom{j}{s}}{2\binom{j}{r}}$ strictly decrease as j increases on the semi-interval $r \leq j < q/2$, moreover, for even q, all these numbers are bigger than $\frac{\binom{q/2}{s}}{\binom{q/2}{r}}$ (see Lemma 8.7 for function $\psi(m)$). It now remains to use Lemma 9.7. The numbers D_i strictly decrease as i increases on the segment $s \leq i \leq r$. Indeed, as i increases, the numerator

$$\sum_{i \leq k \leq (q-i)} \binom{\min(k, q - k)}{s}$$

strictly decreases, whereas the denominator

$$\sum_{i \leq k \leq (q-i)} \binom{\min(k, q - k)}{r} = \sum_{r \leq k \leq (q-r)} \binom{\min(k, q - k)}{r}$$

remains unchanged.

2) It is readily seen that

$$D_0 = \cdots = D_s = \frac{\sum\limits_{s \leq k \leq (q-s)} \binom{\min(k,q-k)}{s}}{\sum\limits_{r \leq k \leq (q-r)} \binom{\min(k,q-k)}{r}}.$$

Formula (6) for numbers $D_0 = \cdots = D_s$ follows from Lemma 9.4. Part 1) shows that this formula gives the maximal value of the numbers D_i. The lemma is thus proved. □

We are now ready to prove Theorem 9.3.

Proof of Theorem 9.3 1) According to Lemma 9.6, the function we maximize can be written in the form

$$\frac{F_r(h)}{\Phi_r(h)} = \frac{\sum\limits_{0 \leq i \leq [q/2]} (\Delta h)_i \sum\limits_{i \leq k \leq q-i} \binom{k}{r}}{\sum\limits_{0 \leq i \leq [q/2]} (\Delta h)_i \sum\limits_{i \leq k \leq (q-i)} \binom{\min(k,q-k)}{r}}.$$

On the simplex under consideration, all numbers $(\Delta h)_i$ are nonnegative. Using Lemma 9.8 and Lemma 9.7, we see that the maximum of the function is attained for $(\Delta h)_0 > 0, (\Delta h)_1 = \cdots = (\Delta h)_{[q/2]} = 0$. This means that the maximum is equal to D_0 and is attained for $h_0 > 0, h_0 = \cdots = h_q$.

2) According to Lemma 9.6, the function we maximize can be written in the form

$$\frac{\Phi_s(h)}{\Phi_r(h)} = \frac{\sum\limits_{0 \leq i \leq [q/2]} (\Delta h)_i \sum\limits_{i \leq k \leq (q-i)} \binom{\min(k,q-k)}{s}}{\sum\limits_{0 \leq i \leq [q/2]} (\Delta h)_i \sum\limits_{i \leq k \leq (q-i)} \binom{\min(k,q-k)}{r}}.$$

On the simplex under consideration, all numbers $(\Delta h)_i$ are nonnegative. Using Lemma 9.9 and Lemma 9.7, we obtain that the maximum of the function is attained for $(\Delta h)_0 + \cdots + (\Delta h)_s > 0, (\Delta h)_{s+1} = \cdots = (\Delta h)_{[q/2]} = 0$. This means that the

maximum is equal to $D_0 = \cdots = D_s$ and is attained for $0 \le h_s, h_s = \cdots = h_{q-s}$. Theorem 9.3 and Theorem 9.1 are thus proved. $\qquad\square$

10 A generalization of Nikulin's Theorem

The problem of estimating the average number of l-dimensional faces on k-dimensional faces of a convex n-dimensional polytope, not necessarily simple, can be reduced to a series of problems on a possible mutual disposition of an $(n-1)$-dimensional convex polytope and a hyperplane. For such reduction, we need to consider a generic linear function on the polytope and, in the spirit of Morse theory, to study the level hypersurfaces of this linear function passing through vertices of the polytope.

Let Δ be a convex n-dimensional polytope, not necessarily simple. Denote by $f_{l,k}$ the number of all pairs consisting of an l-dimensional face of the polytope Δ and a k-dimensional face containing it. The average number of l-dimensional faces on k-dimensional faces of the polytope Δ is the ratio $f_{l,k}/f_k$, where f_k is the number of k-dimensional faces of the polytope. Let us show how to reduce the problem of estimating this number to an $(n-1)$-dimensional problem.

Fix a generic linear function L. With each vertex A of the polytope Δ, associate the pair consisting of a polytope $\Delta(A)$ and its hyperplane section L_A. This pair is defined up to a projective transformation. Here is the definition of this pair. Near each vertex A, the polytope Δ looks like a convex n-dimensional cone. This cone is sectioned by the hyperplane defined by the equation $L(x) = L(A)$. The pair $\Delta(A), L_A$ is defined as the projectivization of the pair consisting of the cone and the hyperplane described above.

In the theorem on reduction stated below, we will also assume that the polytope Δ is simple at the edges. The theorem is valid even without this assumption. However, it helps to shorten the proof, and in the sequel we will not need polytopes that are not simple at the edges.

Denote by $f_k(\Delta(A))$ the number of all k-dimensional faces of the polytope $\Delta(A)$, by $f_k(\Delta(A), L_A)$ the number of all k-dimensional faces of the polytope $\Delta(A)$ disjoint from the hyperplane L_A, and by $f_{l,k}(\Delta(A), L_A)$ the number of all pairs consisting of an l-dimensional face of the polytope $\Delta(A)$ disjoint from the hyperplane L_A and any k-dimensional face containing it. The set of all vertices of the polytope Δ will be denoted by V.

Theorem on reduction *Let Δ be a polytope simple at the edges, and L a generic linear function on it. Then we have the following inequalities:*

1. *for $1 < k \le (n+1)/2$,*

$$\frac{f_{0,k}}{f_k} \le \max_{A \in V} \frac{2f_{k-1}(\Delta(A))}{f_{k-1}(\Delta(A), L_A)},$$

2. *for $0 < l < k \le (n+1)/2$,*

$$\frac{f_{l,k}}{f_k} \le \max_{A \in V} \frac{f_{l-1,k-1}(\Delta(A), L_A)}{f_{k-1}(\Delta(A), L_A)}.$$

Proof 1) The numerator $f_{0,k}$, as well as the denominator f_k, of the ratio $f_{0,k}/f_k$ are representable as sums of nonnegative numbers over the vertices of the polytope Δ. To this end, to each pair $\Gamma_0 \in \Gamma_k$ consisting of a vertex Γ_0 and a k-dimensional face Γ_k containing it, assign the vertex Γ_0. To each k-dimensional face

Γ_k assign the two vertices, where the function L restricted to the face Γ_k attains its maximum and minimum.

Summing up the associated objects over the vertices, we obtain :

$$f_{0,k} = \sum_{A \in V} f_{k-1}(\Delta(A)),$$

$$f_k = \sum_{A \in V} \frac{1}{2} f_{k-1}(\Delta(A), L_A).$$

The number $f_{k-1}(\Delta(A), L_A)$ is strictly positive for each vertex A, since for $k \leq (n+1)/2$, a generic hyperplane does not intersect at least one $(k-1)$-dimensional face of the n-dimensional polytope $\Delta(A)$ (see Corollary 4.5). To conclude the proof of part 1), it suffices to use Lemma 2.1.

2) Analogously to part 1), the numerator $f_{l,k}$, as well as the denominator f_k, of the ratio $f_{l,k}/f_k$ are representable as sums of nonnegative numbers over the vertices of the polytope Δ. To this end, with each pair $\Gamma_l \subset \Gamma_k$ consisting of a face Γ_l and a k-dimensional face Γ_k containing it, associate two vertices, where the function L restricted to the face Γ_l attains the maximum and the minimum. With each k-dimensional face Γ_k, associate two vertices, where the function L restricted to the face Γ_k attains the maximum and the minimum. Summing up the associated objects over all vertices, we obtain:

$$f_{l,k} = \sum_{A \in V} \frac{1}{2} f_{l-1,k-1}(\Delta(A), L_A),$$

$$f_k = \sum_{A \in V} \frac{1}{2} f_{k-1}(\Delta(A), L_A).$$

To conclude the proof of part 2), it suffices to use Lemma 2.1.   □

We are now ready to prove the central result of the article, which generalizes Nikulin's estimate to the case of polytopes simple at the edges.

Theorem *The average number of l-dimensional faces on k-dimensional faces of an n-dimensional polytope simple at the edges is strictly less than*

$$\binom{n-l}{n-k} \frac{\binom{[n/2]}{l} + \binom{[(n+1)/2]}{l}}{\binom{[n/2]}{k} + \binom{[(n+1)/2]}{k}}.$$

for $0 \leq l < k \leq (n+1)/2$, $1 < k$.

Proof A slightly weaker result, where the strict inequalities are replaced with non-strict inequalities, is a direct corollary of the theorem on reduction and Theorem 9.4, which allows to estimate the numbers $\frac{f_{k-1}(\Delta(A))}{f_{k-1}(\Delta(A), L_A)}$ and $\frac{f_{l-1,k-1}(\Delta(A), L_A)}{f_{k-1}(\Delta(A), L_A)}$.

It remains to explain why the inequalities are strict. The point is that among the vertices of the polytope Δ there are two vertices, where the function L attains the maximal and the minimal values. Sections L_A of polytopes $\Delta(A)$ are certainly not successful for these vertices, since the hyperplane L_A does not intersect the polytope $\Delta(A)$. Hence the ratios from Theorem 9.4 are certainly not extremal for these vertices. Adding this remark to the proof of the theorem on reduction, we obtain that all inequalities are in fact strict.   □

To conclude, let us revisit classical Theorem 1.1, according to which the average number of edges on faces of a 3-polytope is strictly less than 6. The theorem on reduction reduces the problem of estimating this number to the following simple two-dimensional problem: estimate from above the ratio of the number of pairs consisting of a vertex of a polygon and an edge containing this vertex, to the number of edges of the polygon disjoint from a generic line.

If the polygon has m edges, then the number of pairs is $2m$, and the number of edges disjoint from a generic line is m, if the line does not intersect the polygon, and is $(m-2)$ in the opposite case. The desired ratio is either $2m/m = 2$ or $2m/(m-2)$. Since the number of edges of a polygon is at least three, the ratio does not exceed 6, which proves Theorem 1.1 (note that in accordance with Theorem 9.1, the maximal value of the desired ratio is attained in the case of triangle intersected by the line). The argument just given does not even use the Euler formula and is most likely the simplest proof of Theorem 1.1.

References

[1] Coxeter, H. X. M., *Discrete groups generated by reflections* Ann. Math. **35** (1934), No. 3, pp. 588–621

[2] Vinberg, E.B., *Discrete groups generated by reflections in Lobachevsky spaces (Russian)* Matem. Sb. **72** (1967), No. 3, pp. 471–488

[3] Khovanskii, A.G. *Hyperplane sections of polyhedra, toric varieties and discrete groups in Lobachevsky space* Funct. Anal. and its Appl. **20** (1986) No. 1 pp. 41–50

[4] Prokhorov, M.N. *Absence of discrete groups of reflections with a noncompact fundamental polyhedron of finite volume in a Lobachevsky space of high dimension. (Russian)* Izv. AN SSSR, ser. mat. **50** (1986) No. 2. pp. 320–332

[5] Vinberg, E.B. *The nonexistence of crystallographic reflection groups in Lobachevsky spaces of large dimension. (Russian)* Funktsion. analiz i ego pril. **15** (1981) No. 2, pp. 67–68

[6] Vinberg, E.B. *The nonexistence of crystallographic reflection groups in Lobachevsky spaces of large dimension. (Russian)* Trudy MMO **47** (1984) pp. 68–102

[7] Nikulin, V.V. *On the arithmetic groups generated by reflections in Lobachevsky spaces. (Russian)* Izv. AN SSSR, ser. mat. **44** (1980) pp. 637–699

[8] Nikulin, V.V. *On the classification of arithmetic groups generated by reflections in Lobachevsky spaces. (Russian)* Izv. AN SSSR, ser. mat. **45** (1981) pp. 113–142

[9] McMullen, P. *On simple polytopes* Invent. math. **113** (1993) pp. 419–444

[10] Timorin, V.A. *An analogue of the Hodge–Riemann relations for simple polytopes* Russian Mathematical Surveys **54** (1999) No. 2 pp. 381–426

[11] Timorin V.A. *On polytopes that are simple at the edges* Funct. Anal. and its Appl. **35** (2001) No. 3 pp. 189–198

[12] Sommerville, D.M.Y. *The relations connecting the angle-sums and volume of polytope in space of n dimensions* Proc. Roy. Soc. London, ser. A115 (1927) pp. 103–119

[13] McMullen, P. *The numbers of faces of simplicial polytopes* Israel J. Math. **9** (1971) pp. 559–570

[14] Stanley, R.P. *The number of faces of a simplicial convex polytope* Adv. Math. **35** (1980) pp. 236–238

[15] Billera, L. J.; Lee, C. W. *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes* J. Comb. Theory, Ser. A **31** (1981) pp. 237–255

[16] Macaulay, F.S. *Some properties of enumeration in the theory of modular systems* Proc. London Math. Soc. **26** (1927) pp. 531–555

[17] Atiyah, M.F. *Convexity and commuting Hamiltonians* Bull. London Math. Soc. **14** (1982) pp. 1–15

[18] Brondsted A. *An introduction to convex polytopes* Springer-Verlag 1983

[19] McMullen P. *The maximum numbers of faces of a convex polytope* Mathematika **17** (1970) pp. 179–184

[20] Bondesen A., Brondsted A. *A dual proof of the upper bound conjecture for convex polytopes* Math. Scand. **46** (1980) pp. 95–102

[21] McMullen P., Shephard G.C. *Convex polytopes and the upper bound conjecture* London Mathematical Society Lecture Note Series, Cambridge University Press, London–New York **3** (1971)